Stochastic and local volatility benchmark problems

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1 SABR stochastic-local volatility model

The *Stochastic Alpha Beta Rho (SABR)* model [6] is an established SDE system which is often used for interest rates and FX modeling in practice. The SABR model is based on a toletric local volatility component in terms of a model toleter, β. The formal definition of the SABR model reads

$$
\begin{aligned}
\mathrm{d}S(t) &= \sigma(t)S^{\beta}(t)\mathrm{d}W_S(t), & S(0) &= S_0 \exp\left(rT\right),\\
\mathrm{d}\sigma(t) &= \alpha\sigma(t)\mathrm{d}W_\sigma(t), & \sigma(0) &= \sigma_0.
\end{aligned}
$$

where $S(t) = \bar{S}(t)\exp\left(r(T-t)\right)$ denotes the forward value of the underlying asset $\bar{S}(t)$, with r the interest rate, S_0 the spot price and T the contract's final time. Quantity $\sigma(t)$ denotes the stochastic volatility, $W_S(t)$ and $W_\sigma(t)$ are two correlated Brownian motions with constant correlation coefficient ρ (i.e. $W_S W_\sigma = \rho t$). The open model toleters are $\alpha > 0$ (the volatility of the volatility), $0 \leq \beta \leq 1$ (the elasticity) and ρ (the correlation coefficient). The corresponding PDE for the valuation of options is given by:

$$
\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 u}{\partial S^2} + \rho\alpha S^{\beta}\sigma^2\frac{\partial^2 u}{\partial\sigma\partial S} + \frac{1}{2}\alpha^2\sigma^2\frac{\partial^2 u}{\partial\sigma^2} - ru = 0
$$

for $S > 0$, $\sigma > 0$ and $0 \leq t < T$.

Two toleter sets:

- Set I ([4]): $T = 2$, $r = 0.0$, $S_0 = 0.5$, $\sigma_0 = 0.5$, $\alpha = 0.4$, $\beta = 0.5$ and $\rho = 0$.
- Set II ([2]): $T = 10$, $r = 0.0$, $S_0 = 0.07$, $\sigma_0 = 0.4$, $\alpha = 0.8$, $\beta = 0.5$ and $\rho = -0.6$.

European call option payoff $\max(S(T) - K_i(T), 0)$ with three strikes

$$
\begin{aligned}
K_i(T) &= S(0)\exp(0.1 \times \sqrt{T} \times \delta_i),\\
\delta_i &= -1.0, 0.0, 1.0.
\end{aligned}
$$

- Output: u for three strikes and two toleter sets.

- Benchmark: Error in the solution as a function of CPU-time.
- Headings:

```
[U]=SABReuCallI_$MTH(tol)
[U]=SABReuCallII_$MTH(tol)
```

`$MTH` should be replaced with a three/four/five-letter method-specific code.
`tol` controls the computational effort and consequently the accuracy and CPU-time.
The absolute error in the computed solution should be in the order of `tol`.
`U` should be a row vector with three elements `(U(1) U(2) U(3))`.

Notes:

- Consider implied volatilities next to option prices, for comparison purposes.
- For $\rho = 0$, there is a formula for the exact simulation of the SABR model [7].
- The use of time discretization MC schemes can give a loss of the martingale property. A correction must be introduced then.

2 Quadratic local stochastic volatility model

In the following τ denotes forward time and t backward time.

See e.g. [8]:

$$\begin{cases} dS_\tau &= rS_\tau \, d\tau + \sqrt{V_\tau}\, f(S_\tau)\, dW^1_\tau, \\ dV_\tau &= \kappa(\eta - V_\tau)\, d\tau + \sigma\sqrt{V_\tau}\, dW^2_\tau, \end{cases}$$

with $f(s) = \frac{1}{2}\alpha s^2 + \beta s + \gamma$. Select

- Heston: $\alpha = 0$, $\beta = 1$, $\gamma = 0$.
- QLSV: $\alpha = 0.02$, $\beta = 0$, $\gamma = 0$.

PDE:

$$\frac{\partial u}{\partial t} + \tfrac{1}{2} f(s)^2 v \frac{\partial^2 u}{\partial s^2} + \rho\sigma f(s) v \frac{\partial^2 u}{\partial s \partial v} + \tfrac{1}{2}\sigma^2 v \frac{\partial^2 u}{\partial v^2} + rs\frac{\partial u}{\partial s} + \kappa(\eta - v)\frac{\partial u}{\partial v} - ru = 0$$

for $s > 0$, $v > 0$ and $0 < t \leq T$.

One toleter set (see [8]):

$$T = 1,\ r = 0,\ \kappa = 2.58,\ \eta = 0.043,\ \sigma = 1,\ \rho = -0.36.$$

Consider

- European call option payoff $\max(s-K,0)$ with $K=100$.
- Double-no-touch option paying 1 if $L < S_\tau < U$ (for all τ) and 0 else with $L=50$, $U=150$.

Three spot values: $(S_0, V_0) = (S_0, 0.114)$ for $S_0 = 75, 100, 125$.

- Output: u for three spot values, two models (Heston and QLSV) and two pay-offs (European call and Double-no-touch).
- Benchmark: Error in the solution as a function of CPU-time.
- Headings:

```
[U]=HSTeuCall_$MTH(tol)
[U]=HSTdnTouch_$MTH(tol)
[U]=QLSVeuCall_$MTH(tol)
[U]=QLSVdnTouch_$MTH(tol)
```

  `$MTH` should be replaced with a three/four/five-letter method-specific code.
  `tol` controls the computational effort and consequently the accuracy and CPU-time.
  The absolute error in the computed solution should be in the order of `tol`.
  `U` should be a row vector with three elements `(U(1) U(2) U(3))`.

Notes:

- Feller condition is violated.
- If $\alpha=0$, $\beta=1$, $\gamma=0$, $\rho=0$ there are semi-closed analytic formulas for both options.

3 Heston–Hull–White model

The Heston–Hull–White model is a hybrid asset price model combining the Heston stochastic volatility and Hull–White stochastic interest rate models, see e.g. [3, 5].

HHW SDE:

$$\begin{cases} dS_\tau &= R_\tau S_\tau \, d\tau + \sqrt{V_\tau}\, S_\tau \, dW_\tau^1, \\ dV_\tau &= \kappa(\eta - V_\tau)\, d\tau + \sigma_1 \sqrt{V_\tau}\, dW_\tau^2, \\ dR_\tau &= a(b(\tau) - R_\tau)\, d\tau + \sigma_2 \, dW_\tau^3. \end{cases}$$

HHW PDE:

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}s^2 v\frac{\partial^2 u}{\partial s^2} + \tfrac{1}{2}\sigma_1^2 v\frac{\partial^2 u}{\partial v^2} + \tfrac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial r^2}$$
$$+ \rho_{12}\sigma_1 sv\frac{\partial^2 u}{\partial s\partial v} + \rho_{13}\sigma_2 s\sqrt{v}\frac{\partial^2 u}{\partial s\partial r} + \rho_{23}\sigma_1\sigma_2\sqrt{v}\frac{\partial^2 u}{\partial v\partial r}$$
$$+ rs\frac{\partial u}{\partial s} + \kappa(\eta - v)\frac{\partial u}{\partial v} + a(b(T-t) - r)\frac{\partial u}{\partial r} - ru = 0$$

for $s > 0$, $v > 0$, $-\infty < r < \infty$ and $0 < t \le T$.

Two toleter sets (cf. [1, 5]):

$T = 10,\ \kappa = 0.5,\ \eta = 0.04,\ \sigma_1 = 1,\ \sigma_2 = 0.09,\ \rho_{12} = -0.9,\ \rho_{13} = 0.6\ (0),\ \rho_{23} = -0.7\ (0),$
$a = 0.08$ and $b(\tau) \equiv 0.10$.

European call option payoff $\max(s - K, 0)$ with $K = 100$.

Three spot values: $(S_0, V_0, R_0) = (S_0, 0.04, 0.10)$ for $S_0 = 75, 100, 125$.

- Output: u for three spot values and two toleter sets.
- Benchmark: Error in the solution as a function of CPU-time.
- Headings:

```
[U]=HHWeuCallI_$MTH(tol)
[U]=HHWeuCallII_$MTH(tol)
```

  `$MTH` should be replaced with a three/four/five-letter method-specific code.
  `tol` controls the computational effort and consequently the accuracy and CPU-time.
  The absolute error in the computed solution should be in the order of `tol`.
  `U` should be a row vector with three elements `(U(1) U(2) U(3))`.

Notes:

- Feller condition is violated.
- For $\rho_{13} = \rho_{23} = 0$ there is semi-closed analytic formula akin to Heston.
- Transformation to 2D PDE with time-dependent coefficients is possible. Hence, if numerical PDE approach is followed, indicate which PDE is solved (2D or 3D).

References

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